Locally Anisotropic Interactions: II. Torsions and Curvatures of Higher Order Anisotropic Superspaces

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Abstract

Torsions, curvatures, structure equations and Bianchi identities for locally anisotropic superspaces (containing as particular cases different supersymmetric extensions and prolongations of Riemann, Finsler, Lagrange and Kaluza–Klein spaces) are investigated.

1 Introduction

Locally anisotropic superspaces are modeled as vector superbundles (vs–bundles) provided with compatible nonlinear and distinguished connections (in brief, N– and d–connections) and metric structures [28]. Higher order anisotropies are introduced on higher order tangent superbundles or on corresponding generalizations of vs–bundles called distinguished vector superbundles bundles, dvs–bundles (see for details the first companion of this work[29]). The aim of the current paper is to continue our investigations of basic geometric structures on higher dimension superspaces with generic local anisotropy and to explore possible applications in theoretical and mathematical physics [27, 30, 28]. In order to compute torsions, curvatures in dvs–bundles as well to define components of Bianchi identities and Cartan structure equations we shall propose a supersymmetric variant of the differential geometric technique developed in [19, 20, 21].

Section 2 contains the basic definitions on nonlinear connection structures in locally anisotropic superspaces after [29]. The properties of fundamental

distinguished geometric objects such as d–tensors and d–connections and distinguished torsions and curvatures are correspondingly considered in sections 3 and 4. Bianchi and Ricci identities as well Cartan structure equations are analyzed in sections 5 and 6. The problem of compatibility of N–connection, d–connection and metric structures is studied in section 7. Section 8 is devoted to the geometry of higher order tangent superbundles. In section 9 we present some possible variants of supersymmetric extensions of Finsler spaces. Higher order prolongations of Finsler and Lagrange superspaces as well higher order Lagrange superspaces are described in section 10. Finally, in section 11, we draw some conclusions and discussion.

2 Locally Anisotropic Superspaces: Basic Definitions

Let introduce the necessary definitions and denotations on dvs–bundles [29, 28] (on supermanifolds and superbundles see, for instance, [32, 24, 8]):

Locally a supermanifold (superspace, s–space) $\widetilde{M}$ has some commuting $\widehat{x}^i$ (indices $i, j, ...$ run from 1 to n, where n is the even dimension of s–space) and anticommuting coordinates $\theta^{\widehat{i}}$(indices $\widehat{i}, \widehat{j}, ... = 1, 2, ..., l$, where l is the odd dimension of s–space). We shall use coordinates x^I provided with general indices $I, J, ...$ of type $I = (i, \widehat{i}), J = (j, \widehat{j}), ...$ By $\widehat{\mathcal{E}}$ we denote a vector superspace (vs–space) of dimension (m, l) . With respect to a chosen frame we parametrize an element $y \in \widehat{\mathcal{E}}$ as $y = (\widehat{y}, \zeta) = \{y^A = \left(\widehat{y}^a, \zeta^{\widehat{a}}\right)\}$, where $a = 1, 2, ..., m$ and $\widehat{a} = 1, 2, ..., l$. Indices of type $A = (a, \widehat{a}), B = (b, \widehat{b}), ...$ will parametrize objects on vs–spaces.

A distinguished vector superbundle (dvs–bundle) $\widetilde{\mathcal{E}}^{<z>} = (\tilde{E}^{<z>}, \pi^{<d>}, \mathcal{F}^{<d>}, \tilde{M})$, with surjective projection $\pi^{<z>} : \tilde{E}^{<z>} \to \tilde{M}$, where $\tilde{M}$ and $\tilde{E}^{<z>}$ are respectively base and total s–spaces and the dv–space $\mathcal{F}^{<z>}$ is the standard fibre, can be defined [29] in a usual manner (see [8, 11, 5, 31], on vector superbundles, and [15, 20, 19] on vector bundles).

The typical fibre $\mathcal{F}^{<z>}$ is a distinguished vector superspace (dvs–space) of dimension (m, l) constructed as an oriented direct sum $\mathcal{F}^{<z>} = \mathcal{F}_{(1)} \oplus \mathcal{F}_{(2)} \oplus ... \oplus \mathcal{F}_{(z)}$ of vs–spaces $\mathcal{F}_{(p)}, \dim \mathcal{F}_{(p)} = (m_{(p)}, l_{(p)})$, where $(p) = (1), (2), ..., (z)$, $\sum_{p=1}^{p=z} m_{(p)} = m, \sum_{p=1}^{p=z} l_{(p)} = l$.

Coordinates on $\mathcal{F}^{<p>}$ are denoted as

$$y^{<p>} = (y_{(1)}, y_{(2)}, ..., y_{(p)}) = (\hat{y}_{(1)}, \zeta_{(1)}, \hat{y}_{(2)}, \zeta_{(2)}, ..., \hat{y}_{(p)}, \zeta_{(p)}) =$$

$$\{y^{<A>} = (\hat{y}^{<a>}, \zeta^{<\hat{a}>}) = (\hat{y}^{<a>}, y^{<\hat{a}>})\},$$

where bracketed indices are correspondingly split on $\mathcal{F}_{(p)}$–components:

$$< A >= \left(A_{(1)}, A_{(2)}, ..., A_{(p)}\right), \ < a >= (a_{(1)}, a_{(2)}, ..., a_{(p)})$$

$$\text{and } < \widehat{a} >= (\widehat{a}_{(1)}\widehat{a}_{(2)}, ..., \widehat{a}_{(p)}),$$

We shall also write indices in a more simplified form, $< A >= (A_1, A_2, ..., A_p)$, $< a >= (a_1, a_2, ..., a_p)$ and $< \widehat{a} >= (\widehat{a}_1, \widehat{a}_2, ..., \widehat{a}_p)$ if this will give not rise to ambiguities.

A dvs–bundle $\widetilde{\mathcal{E}}^{<z>}$ can be considered as an oriented set of vs–bundles $\pi^{<p>}$: $\tilde{E}^{<p>} \rightarrow \tilde{E}^{<p-1>}$ (with typical fibers $\mathcal{F}^{<p>}, p = 1, 2, ..., z$); $\tilde{E}^{<0>} = \tilde{M}$. The index z (p) denotes the total (intermediate) numbers of consequent vs–bundle coverings of $\tilde{M}$. Local coordinates on $\widetilde{\mathcal{E}}^{<p>}$ are parametrized in this manner:

$$u_{(p)} = (x, y_{<p>}) = (x, y_{(1)}, y_{(2)}, ..., y_{(p)}) = (\hat{x}, \theta, \hat{y}_{<p>}, \zeta_{<p>}) =$$

$$(\hat{x}, \theta, \hat{y}_{(1)}, \zeta_{(1)}, \hat{y}_{(2)}, \zeta_{(2)}, ..., \hat{y}_{(p)}, \zeta_{(p)}) = \{u^{<\alpha>} = (x^I, y^{<A>}) = (\hat{x}^i, \theta^{\hat{i}}, \hat{y}^{<a>}, \zeta^{<\hat{a}>})$$

$$= (\hat{x}^i, x^{\hat{i}}, \hat{y}^{<a>}, y^{<\hat{a}>})\} = (x^I, y^{A_1}, y^{A_2}, ..., y^{A_p}) = ...$$

The coordinate transforms on dvs–bundles $\{u^{<\alpha>} = (x^I, y^{<A>})\} \rightarrow \{u^{<\alpha'>} = (x^{I'} = x^{I'}(x^I),\ y^{<A'>} = K^{<A'>}_{<A>} y^{<A>})\}$ are given by recurrent maps:

$$x^{I'} = x^{I'}(x^I), \quad srank(\frac{\partial x^{I'}}{\partial x^I}) = (n, k), \tag{1}$$

$$y^{A'_1}_{(1)} = K^{A'_1}_{A_1}(x) y^{A_1}_{(1)}, ..., y^{A'_p}_{(p)} = K^{A'_p}_{A_p}(u_{(p-1)}) y^{A_p}_{(p)}, ..., y^{A'_z}_{(z)} = K^{A'_z}_{A_z}(u_{(z-1)}) y^{A_z}_{(z)},$$

where $K^{A'_p}_{A_p}(u_{(p-1)}) \in G(m_{(p)}, l_{(p)}, \Lambda), \Lambda$ is the Grassman algebra (provided with both structures of a Banach algebra and Euclidean topological space) used for construction of our superspaces. A nonlinear connection (N–connection) in a dvs–bundle $\widetilde{\mathcal{E}}^{<z>}$ can be introduced [29] as a regular supersymmetric distribution $N\left(\widetilde{\mathcal{E}}^{<z>}\right)$ (horizontal distribution being supplementary to the vertical s-distribution $V\left(\widetilde{\mathcal{E}}^{<z>}\right)$) determined by maps $N : u \in \widetilde{\mathcal{E}}^{<z>} \rightarrow N(u) \subset T_u\left(\widetilde{\mathcal{E}}^{<z>}\right)$ for which one holds the Whitney sum:

$$T\left(\widetilde{\mathcal{E}}^{<z>}\right) = N\left(\widetilde{\mathcal{E}}^{<z>}\right) \oplus V\left(\widetilde{\mathcal{E}}^{<z>}\right). \tag{2}$$

Locally a N–connection in $\widetilde{\mathcal{E}}^{<z>}$ is given by its coefficients

$$N^{A_1}_{(01)I}(u), (N^{A_2}_{(02)I}(u), N^{A_2}_{(12)A_1}(u)), ..., (N^{A_p}_{(0p)I}(u), N^{A_p}_{(1p)A_1}(u), ...N^{A_p}_{(p-1p)A_{p-1}}(u)), ...,$$

$$(N^{A_z}_{(0z)I}(u), N^{A_z}_{(1z)A_1}(u), ..., N^{A_z}_{(pz)A_p}(u), ..., N^{A_z}_{(z-1z)A_{z-1}}(u)).$$

If a N–connection structure is defined we must extend the operation of partial derivation in this manner:

$$\delta_{\bullet}=\widehat{\mathbf{N}}(u)\times\partial_{\bullet}, \tag{3}$$

where column matrices

$$\delta_{\bullet}=\delta_{<\alpha>}=\begin{pmatrix}\delta_I\\ \delta_{A_1}\\ \delta_{A_2}\\ ...\\ \delta_{A_z}\end{pmatrix}=\begin{pmatrix}\frac{\delta}{\partial x^I}\\ \frac{\delta}{\partial y^{A_1}_{(1)}}\\ \frac{\delta}{\partial y^{A_2}_{(2)}}\\ ...\\ \frac{\delta}{\partial y^{A_z}_{(z)}}\end{pmatrix} \text{ and } \partial_{\bullet}=\partial_{<\alpha>}=\begin{pmatrix}\partial_I\\ \partial_{A_1}\\ \partial_{A_2}\\ ...\\ \partial_{A_z}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x^I}\\ \frac{\partial}{\partial y^{A_1}_{(1)}}\\ \frac{\partial}{\partial y^{A_2}_{(2)}}\\ ...\\ \frac{\partial}{\partial y^{A_z}_{(z)}}\end{pmatrix}$$

defines respectively a locally adapted basis (frame) and a usual coordinate frame and matrix

$$\widehat{\mathbf{N}}=\begin{pmatrix}1 & -N_I^{A_1} & -N_I^{A_2} & ... & -N_I^{A_z}\\ 0 & 1 & -N_{A_1}^{A_2} & ... & -N_{A_1}^{A_z}\\ 0 & 0 & 1 & ... & -N_{A_2}^{A_z}\\ ... & ... & ... & ... & ...\\ 0 & 0 & 0 & ... & 1\end{pmatrix}$$

is constructed by using components of N–connection.

The dual to (3) bases are denoted correspondingly as

$$\delta^{\bullet}=\mathbf{d}^{\bullet}\times\mathbf{M}(u), \tag{4}$$

where

$$\delta^{\bullet}=\begin{pmatrix}\delta x^I & \delta y^{A_1} & \delta y^{A_2} & ... & \delta y^{A_z}\end{pmatrix},\ \mathbf{d}^{\bullet}=\begin{pmatrix}dx^I & dy^{A_1} & \delta y^{A_2} & ... & \delta y^{A_z}\end{pmatrix}$$

and matrix

$$\mathbf{M}=\begin{pmatrix}1 & M^{A_1}_{(1)I} & M^{A_2}_{(2)I} & ... & M^{A_z}_{(z)I}\\ 0 & 1 & M^{A_2}_{(2)A_1} & ... & M^{A_z}_{(z)A_1}\\ 0 & 0 & 1 & ... & M^{A_z}_{(z)A_2}\\ ... & ... & ... & ... & ...\\ 0 & 0 & 0 & ... & 1\end{pmatrix},$$

defining the dual components of the N–connection, is s–inverse to matrix $\mathbf{N}$ from (3).

With respect to coordinate changes (1) the la-base (3) transforms as

$$\frac{\delta}{\partial x^I}=\frac{\partial x^{I'}}{\partial x^I}\frac{\delta}{\partial x^{I'}},\ \frac{\delta}{\partial y^{A_p}_{(p)}}=K^{A'_p}_{A_p}\frac{\delta}{\partial y^{A'_p}_{(p)}},\forall p=1,2,...,z, \tag{5}$$

where $K_I^{I'} = \frac{\partial x^{I'}}{\partial x^I}$.

We obtain a supersymmetric generalization of the Miron–Atanasiu [21] osculator bundle $\left(Osc^z \tilde{M}, \pi, \tilde{M}\right)$ if the fiber space is taken to be a direct sum of z vector s-spaces of the same dimension $\dim \mathcal{F} = \dim \widetilde{M}$, i.e. $\mathcal{F}^{<d>} = \mathcal{F} \oplus \mathcal{F} \oplus ... \oplus \mathcal{F}$. For $z = 1$ the $Osc^1 \widetilde{M}$ is the tangent s-bundle $T\widetilde{M}$. We also note that for the osculator s-bundle $\left(Osc^z \tilde{M}, \pi, \tilde{M}\right)$ can be defined an additional s–tangent structure $J : \Xi\left(Osc^z \tilde{M}\right) \to \Xi\left(Osc^z \tilde{M}\right)$ defined as

$$\frac{\delta}{\partial y^I_{(1)}} = J\left(\frac{\delta}{\partial x^I}\right), ..., \frac{\delta}{\partial y^I_{(z-1)}} = J\left(\frac{\delta}{\partial y^I_{(z-2)}}\right), \frac{\partial}{\partial y^I_{(z)}} = J\left(\frac{\delta}{\partial y^I_{(z-1)}}\right). \quad (6)$$

In order to simplify considerations in this work we shall consider only locally trivial vector bundles.

3 Distinguished Tensors and Connections in DVS–Bundles

By using adapted bases (3) and (4) one introduces the algebra $DT(\tilde{\mathcal{E}}^{<z>})$ of distinguished tensor s-fields (ds-fields, ds-tensors, ds-objects) on $\tilde{\mathcal{E}}^{<z>}$, $\mathcal{T} = \mathcal{T}^{pp_1p_2....p_z}_{qq_1q_2...q_z}$, which is equivalent to the tensor algebra of vs-bundle $\pi_{hv_1v_2...v_z} : H\tilde{\mathcal{E}}^{<z>} \oplus V_1\tilde{\mathcal{E}}^{<z>} \oplus V_2\tilde{\mathcal{E}}^{<z>} \oplus ... \oplus V_s\tilde{\mathcal{E}}^{<z>} \to \tilde{\mathcal{E}}^{<z>}$. An element $Q \in \mathcal{T}^{pp_1p_2....p_z}_{qq_1q_2...q_z}$, , ds-field of type $\begin{pmatrix} p & p_1 & p_2 & ... & p_z \\ q & q_1 & q_2 & ... & q_z \end{pmatrix}$, can be written in local form as

$$Q = Q^{I_1...I_pA_1...A_{p_1}E_1...E_{p_2}...D_1....D_{p_s}}_{J_1...J_qB_1...B_{q_1}C_1...C_{q_2}...F_1...F_{q_s}}(u)\delta_{I_1} \otimes \ldots \otimes \delta_{I_p} \otimes d^{J_1} \otimes \ldots \otimes d^{J_q} \otimes$$

$$\delta_{A_1} \otimes \ldots \otimes \delta_{A_{p_1}} \otimes \delta^{B_1}\delta^{B_1} \otimes \ldots \otimes \delta^{B_{q_1}} \otimes \delta_{E_1} \otimes \ldots \otimes \delta_{E_{p_2}} \otimes \delta^{C_1} \otimes \ldots \otimes \delta^{C_{q_2}} \otimes ...$$

$$\otimes \delta_{D_1} \otimes \ldots \otimes \delta_{D_{p_z}} \otimes \delta^{F_1} \otimes \ldots \otimes \delta^{F_{q_z}}. \quad (7)$$

In addition to ds-tensors we can introduce ds-objects with various s-group and coordinate transforms adapted to a global splitting (2).

A linear distinguished connection, d- connection, in dvs-bundle $\tilde{\mathcal{E}}^{<z>}$ is a linear connection D on $\tilde{\mathcal{E}}^{<z>}$ which preserves by parallelism the horizontal and vertical distributions in $\tilde{\mathcal{E}}^{<z>}$.

By a linear connection of a s-manifold we understand a linear connection in its tangent bundle.

Let denote by $\Xi(\tilde{M})$ and $\Xi(\tilde{\mathcal{E}}^{<p>})$, respectively, the modules of vector fields on s-manifold $\tilde{M}$ and dvs-bundle $\tilde{\mathcal{E}}^{<p>}$ and by $\mathcal{F}(\tilde{M})$ and $\mathcal{F}(\tilde{\mathcal{E}}^{<p>})$, respectively, the s-modules of functions on $\tilde{M}$ and on $\tilde{\mathcal{E}}^{<p>}$.

It is clear that for a given global splitting into horizontal and verticals s-subbundles (2) we can associate operators of horizontal and vertical covariant derivations (h- and v-derivations, denoted respectively as $D^{(h)}$ and $D^{(v_1 v_2 \ldots v_z)}$) with properties:

$$D_X Y = (XD)Y = D_{hX}Y + D_{v_1 X}Y + D_{v_2 X}Y + ... + D_{v_z X}Y,$$

where

$$D_X^{(h)}Y = D_{hX}Y, \quad D_X^{(h)}f = (hX)f$$

and

$$D_X^{(v_p)}Y = D_{v_p X}Y, \quad D_X^{(v_p)}f = (v_p X)f, \ (p = 1, ..., z)$$

for every $f \in \mathcal{F}(\tilde{M})$ with decomposition of vectors $X, Y \in \Xi(\tilde{\mathcal{E}}^{<z>})$ into horizontal and vertical parts, $X = hX + v_1 X + ... + v_z X$ and $Y = hY + v_1 Y + ... + v_z Y$.

The local coefficients of a d- connection D in $\tilde{\mathcal{E}}^{<z>}$ with respect to the local adapted frame (3) separate into corresponding distinguished groups. We introduce horizontal local coefficients $(L^I_{JK}, L^{<A>}_{<B>K}) = (L^I{}_{JK}(u), L^{A_1}_{B_1 K}(u), L^{A_2}_{B_2 K}(u), ..., L^{A_z}_{B_z K}(u))$ of $D^{(h)}$ such that

$$D^{(h)}_{(\frac{\delta}{\delta x^K})}\frac{\delta}{\delta x^J} = L^I{}_{JK}(u)\frac{\delta}{\delta x^I}, D^{(h)}_{(\frac{\delta}{\delta x^K})}\frac{\delta}{\delta y^{B_p}_{(p)}} = L^{A_p}_{B_p K}(u)\frac{\delta}{\delta y^{A_p}_{(p)}}, (p = 1, ..., z),$$

$$D^{(h)}_{(\frac{\delta}{\delta x^k})} q = \frac{\delta q}{\delta x^K},$$

and p–vertical local coefficients $(C^I_{J<C>}, C^{<A>}_{<B><C>}) = (C^I_{JC_p}(u), C^{A_1}_{B_1 C_p}(u), C^{A_2}_{B_2 C_p}(u), ..., C^{A_z}_{B_z C_p}(u))$ $(p = 1, ..., z)$ such that

$$D^{(v_p)}_{(\frac{\delta}{\delta y^{C_p}})}\frac{\delta}{\delta x^J} = C^I_{JC_p}(u)\frac{\delta}{\delta x^I}, D^{(v_p)}_{(\frac{\delta}{\delta y^{C_p}})}\frac{\delta}{\delta y^{B_f}_{(f)}} = C^{A_f}_{B_f C_p}\frac{\delta}{\delta y^{A_f}_{(f)}}, D^{(v_p)}_{(\frac{\delta}{\delta y^{C_p}})} q = \frac{\delta q}{\partial y^{C_p}},$$

where $q \in \mathcal{F}(\tilde{\mathcal{E}}^{<z>}),\ f = 1, ..., z$.

The covariant ds-derivation along vector $X = X^I \frac{\delta}{\partial x^I} + Y^{A_1}\frac{\delta}{\partial y^{A_1}} + ... + Y^{A_z}\frac{\partial}{\partial y^{A_z}}$ of a ds-tensor field Q, for instance, of type $\begin{pmatrix} p & p_r \\ q & q_r \end{pmatrix}, 1 \leq r \leq z,$ see (7), can be written as

$$D_X Q = D_X^{(h)}Q + D_X^{(v_1)}Q + ... + D_X^{(v_z)}Q,$$

where h-covariant derivative is defined as

$$D_X^{(h)}Q = X^K Q^{IA_r}_{JB_r|K}\delta_I \otimes \delta_{A_r} \otimes d^J \otimes \delta^{B_r},$$

with components

$$Q^{IA_r}_{JB_r|K} = \frac{\delta Q^{IA_r}_{JB_r}}{\partial x^K} + L^I{}_{HK} Q^{HA_r}_{JB_R} + L^{A_r}_{C_i K} Q^{IC_r}_{JB_i} - L^H{}_{JK} Q^{IA_r}_{HB_r} - L^{C_r}_{B_r K} Q^{IA_r}_{JC_r},$$

and v$_p$-covariant derivatives defined as

$$D^{(v_p)}_X Q = X^{C_p} Q^{IA_r}_{JB_r \perp C_p} \delta_I \otimes \delta_{A_r} \otimes \delta^I \otimes \delta^{B_r},$$

with components

$$Q^{IA_r}_{JB_r \perp C_p} = \frac{\delta Q^{IA_r}_{JB_R}}{\partial y^{C_p}} + C^I{}_{HC_p} Q^{HA_r}_{JB_R} + C^{A_r}_{F_r C_p} Q^{IF_r}_{JB_R} - C^H_{JC_p} Q^{IA_r}_{HF_R} - C^{F_r}_{B_r C_p} Q^{IA_r}_{JF_R}..$$

The above presented formulas show that $D\Gamma = (L, ..., L_{(p)}, ..., C, ..., C_{(p)}, ...)$ are the local coefficients of the d-connection D with respect to the local frame $(\frac{\delta}{\delta x^I}, \frac{\delta}{\partial y^{<A>}})$. If a change (1) of local coordinates on $\tilde{\mathcal{E}}^{<z>}$ is performed we have the following transformation laws of the local coefficients of a d-connection:

$$L^{I'}{}_{J'M'} = \frac{\partial x^{I'}}{\partial x^I} \frac{\partial x^J}{\partial x^{J'}} \frac{\partial x^M}{\partial x^{M'}} L^I{}_{JM} + \frac{\partial x^{I'}}{\partial x^M} \frac{\partial^2 x^M}{\partial x^{J'} \partial x^{M'}}, \tag{8}$$

$$L^{A'_f}_{(f)B'_f M'} = K^{A'_f}_{A_f} K^{B_f}_{B'_f} \frac{\partial x^M}{\partial x^{M'}} \underset{(f)B_f M}{L}^{A_f} + K^{A'_f}_{C_f} \frac{\partial K^{C_f}_{B'_f}}{\partial x^{M'}},$$

....

$$C^{I'}_{(p)J'C'_p} = \frac{\partial x^{I'}}{\partial x^I} \frac{\partial x^J}{\partial x^{J'}} K^{C'_p}_{C_p} C^I_{(p)JC_p}, ..., C^{A'_f}_{B'_f C'_p} = K^{A'_f}_{A_f} K^{B_f}_{B'_f} K^{C_p}_{C'_p} C^{A_f}_{B_f C_p}, ...$$

As in the usual case of tensor calculus on locally isotropic spaces the transformation laws (8) for d-connections differ from those for ds-tensors, which are written (for instance, we consider transformation laws for ds-tensor (7)) as

$$Q^{I'_1 ... I'_p A'_1 ... A'_{p_1} E'_1 ... E'_{p_2} ... D'_1 .... D'_{p_s}}_{J'_1 ... J'_q B'_1 ... B'_{q_1} C'_1 ... C'_{q_2} ... F'_1 ... F'_{q_s}} =$$

$$\frac{\partial x^{I'_1}}{\partial x^{I_1}} \frac{\partial x^{J_1}}{\partial x^{J'_1}} \ldots K^{A'_1}_{A_1} K^{B_1}_{B'_1} \ldots K^{D'_{p_s}}_{D_{p_s}} K^{F_{p_s}}_{F'_{p_s}} Q^{I_1 ... I_p A_1 ... A_{p_1} E_1 ... E_{p_2} ... D_1 .... D_{p_s}}_{J_1 ... J_q B_1 ... B_{q_1} C_1 ... C_{q_2} ... F_1 ... F_{q_s}}.$$

To obtain local formulas on usual higher order anisotropic spaces we have to restrict us with even components of geometric objects by changing, formally, capital indices $(I, J, K, ...)$ into $(i, j, k, a, ..)$ and s-derivation and s-commutation rules into those for real number fields on usual manifolds. We shall consider various applications in the theoretical and mathematical physics of the differential geometry of distinguished vector bundles in our further works.

4 Torsions and Curvatures of D–Connections

Let $\tilde{\mathcal{E}}^{<z>}$ be a dvs–bundle endowed with N-connection and d-connection structures. The torsion of a d-connection is introduced as

$$T(X,Y) = [X, DY\} - [X,Y\}, \quad X, Y \subset \Xi(\tilde{M}).$$

where $[...\}$ is the s–commutator. One holds the following invariant decomposition (by using h– and v–projections associated to N):

$$T(X,Y) = T(hX,hY) + T(hX,v_1Y) + T(v_1X,hX) + T(v_1X,v_1Y) + ...$$

$$+T(v_{p-1}X,v_{p-1}Y) + T(v_{p-1}X,v_pY) + T(v_pX,v_{p-1}X) + T(v_pX,v_pY) + ...$$
$$+T(v_{z-1}X,v_{z-1}Y) + T(v_{z-1}X,v_zY) + T(v_zX,v_{z-1}X) + T(v_zX,v_zY).$$

Taking into account the skewsupersymmetry of T and the equations

$$h[v_pX,v_pY\} = 0, ..., v_f[v_pX,v_pY\} = 0, f \neq p,$$

we can verify that the torsion of a d-connection is completely determined by the following ds-tensor fields:

$$hT(hX,hY) = [X(D^{(h)}h)Y\} - h[hX,hY\}, ...,$$

$$v_pT(hX,hY) = -v_p[hX,hY\}, ...,$$
$$hT(hX,v_pY) = -D_Y^{(v_p)}hX - h[hX,v_pY\}, ...,$$
$$v_pT(hX,v_pY) = D_X^{(h)}v_pY - v_p[hX,v_pY\}, ...,$$
$$v_fT(v_fX,v_fY) = [X(D^{(v_f)}v_f)Y\} - v_f[v_fX,v_fY\}, ...,$$
$$v_pT(v_fX,v_fY) = -v_p[v_fX,v_fY\}, ...,$$
$$v_fT(v_fX,v_pY) = -D_Y^{(v_p)}v_fX - v_f[v_fX,v_pY\}, ...,$$
$$v_pT(v_fX,v_pY) = D_X^{(v_f)}v_pY - v_p[v_fX,v_pY\}, ..., f < p,$$
$$v_{z-1}T(v_{z-1}X,v_{x-1}Y) = [X(D^{(v_{z-1})}v_{z-1})Y\} - v_{z-1}[v_{z-1}X,v_{z-1}Y\}, ...,$$
$$v_zT(v_{z-1}X,v_{z-1}Y) = -v_z[v_{z-1}X,v_{z-1}Y\},$$
$$v_{z-1}T(v_{z-1}X,v_zY) = -D_Y^{(v_z)}v_{z-1}X - v_{z-1}[v_{z-1}X,v_zY\}, ...,$$
$$v_zT(v_{z-1}X,v_zY) = D_X^{(v_{z-1})}v_zY - v_z[v_{z-1}X,v_zY\}.$$

where $X, Y \in \Xi(\tilde{\mathcal{E}}^{<z>})$. In order to get the local form of the ds-tensor fields which determine the torsion of d-connection $D\Gamma$ (the torsions of $D\Gamma$) we use equations

$$[\frac{\delta}{\partial x^J}, \frac{\delta}{\partial x^K}\} = R^{<A>}_{JK}\frac{\delta}{\partial y^{<A>}}, \ [\frac{\delta}{\partial x^J}, \frac{\delta}{\partial y^{<B>}}\} = \frac{\delta N^{<A>}_J}{\partial y^{<B>}}\frac{\delta}{\partial y^{<A>}},$$

where

$$R^{<A>}_{JK} = \frac{\delta N^{<A>}_J}{\partial x^K} - (-)^{|KJ|}\frac{\delta N^{<A>}_K}{\partial x^J},$$

$$(-)^{|K||J|} = (-)^{|KJ|} = (-1)^{|KJ|},$$

$|K| = 0(1)$ for even (odd) components, and introduce notations

$$hT(\frac{\delta}{\delta x^K}, \frac{\delta}{\delta x^J}) = T^I{}_{JK}\frac{\delta}{\delta x^I}, \ v_1T(\frac{\delta}{\delta x^K}, \frac{\delta}{\delta x^J}) = \tilde{T}^{A_1}_{KJ}\frac{\delta}{\partial y^{A_1}}, \tag{9}$$

$$hT(\frac{\delta}{\partial y^{<A>}}, \frac{\delta}{\partial x^J}) = \tilde{P}^I_{J<A>}\frac{\delta}{\delta x^I}, ..., \ v_pT(\frac{\delta}{\partial y^{B_p}}, \frac{\delta}{\delta x^J}) = P^{<A>}_{JB_p}\frac{\delta}{\partial y^{<A>}}, ...,$$

$$v_pT(\frac{\delta}{\partial y^{C_p}}, \frac{\delta}{\partial y^{B_f}}) = S^{<A>}_{B_f C_p}\frac{\delta}{\partial y^{<A>}}.$$

Now we can compute the local components of the torsions, introduced in (9), with respect to a la-frame of a d-connection
$D\Gamma = (L, ..., L_{(p)}, ..., C, ..., C_{(p)}, ...)$:

$$T^I{}_{JK} = L^I{}_{JK} - (-)^{|JK|}L^I{}_{KJ}, \ T^{<A>}_{JK} = R^{<A>}{}_{JK}, \tag{10}$$

$$P^I{}_{J<B>} = C^I{}_{J<B>}, \ P^{<A>}{}_{J<B>} = \frac{\delta N^{<A>}_J}{\partial y^{<B>}} - L^{<A>}{}_{<B>J},$$

$$S^{<A>}{}_{<B><C>} = C^{<A>}{}_{<B><C>} - (-)^{|<B><C>|}C^{<A>}{}_{<C><B>}.$$

The even and odd components of torsions (10) can be specified in explicit form by using decompositions of indices into even and odd parts ($I = (i, \hat{i}), J = (j, \hat{j}), ..$), for instance,

$$T^i{}_{jk} = L^i{}_{jk} - L^i{}_{kj}, \quad T^i{}_{j\hat{k}} = L^i{}_{j\hat{k}} + L^i{}_{\hat{k}j},$$

$$T^{\hat{i}}{}_{jk} = L^{\hat{i}}{}_{jk} - L^{\hat{i}}{}_{kj}, \ldots,$$

and so on (see [19, 20] and [30, 27] explicit formulas on torsions on a la-space M, one omits "tilde" for usual manifolds and bundles).

Another important characteristic of a d-connection $D\Gamma$ is its curvature:

$$R(X, Y)Z = D_{[X}D_{Y\}} - D_{[X,Y\}}Z,$$

where $X,Y,Z \in \Xi(\tilde{E}^{<z>})$. From the properties of h- and v-projections it follows that

$$v_pR(X,Y)hZ = 0, ..., hR(X,Y)v_pZ = 0, v_fR(X,Y)v_pZ = 0, f \neq p, \qquad (11)$$

and

$$R(X,Y)Z = hR(X,Y)hZ + v_1R(X,Y)v_1Z + ... + v_zR(X,Y)v_zZ,$$

where $X,Y,Z \in \Xi(\tilde{E}^{<z>})$. Taking into account properties (11) and the equations

$$R(X,Y) = -(-)^{|XY|}R(Y,X)$$

we prove that the curvature of a d-connection D in the total space of a vs-bundle $\tilde{E}^{<z>}$ is completely determined by the following ds-tensor fields:

$$R(hX,hY)hZ = (D_{[X}^{(h)}D_{Y\}}^{(h)} - D_{[hX,hY\}}^{(h)} \qquad (12)$$

$$-D_{[hX,hY\}}^{(v_1)} - ...D_{[hX,hY\}}^{(v_{z-1})} - D_{[hX,hY\}}^{(v_z)})hZ,$$

$$R(hX,hY)v_pZ = (D_{[X}^{(h)}D_{Y\}}^{(h)} - D_{[hX,hY\}}^{(h)} -$$

$$D_{[hX,hY\}}^{(v_1)})v_pZ - ... - D_{[hX,hY\}}^{(v_{p-1})})v_pZ - D_{[hX,hY\}}^{(v_p)})v_pZ,$$

$$R(v_pX,hY)hZ = (D_{[X}^{(v_p)}D_{Y\}}^{(h)} - D_{[v_pX,hY\}}^{(h)} -$$

$$D_{[v_pX,hY\}}^{(v_1)})hZ - ... - D_{[v_pX,hY\}}^{(v_{p-1})} - D_{[v_pX,hY\}}^{(v_p)})hZ,$$

$$R(v_fX,hY)v_pZ = (D_{[X}^{(v_f)}D_{Y\}}^{(h)} - D_{[v_fX,hY\}}^{(h)} -$$

$$D_{[v_fX,hY\}}^{(v_1)})v_1Z - ... - D_{[v_fX,hY\}}^{(v_{p-1})})v_{p-1}Z - D_{[v_fX,hY\}}^{(v_p)})v_pZ,$$

$$R(v_fX,v_pY)hZ = (D_{[X}^{(v_f)}D_{Y\}}^{(v_p)} - D_{[v_fX,v_pY\}}^{(v_1)})hZ - ...$$

$$-D_{[v_fX,v_pY\}}^{(v_{z-1})} - D_{[v_fX,v_pY\}}^{(v_z)})hZ,$$

$$R(v_fX,v_qY)v_pZ = (D_{[X}^{(v_f)}D_{Y\}}^{(v_q)} - D_{[v_fX,v_qY\}}^{(v_1)})v_1Z - ...$$

$$-D_{[v_fX,v_qY\}}^{(v_{p-1})})v_{p-1}Z - D_{[v_fX,v_qY\}}^{(v_p)})v_pZ,$$

where

$$D_{[X}^{(h)}D_{Y\}}^{(h)} = D_X^{(h)}D_Y^{(h)} - (-)^{|XY|}D_Y^{(h)}D_X^{(h)},$$

$$D_{[X}^{(h)}D_{Y\}}^{(v_p)} = D_X^{(h)}D_Y^{(v_p)} - (-)^{|Xv_pY|}D_Y^{(v_p)}D_X^{(h)},$$

$$D_{[X}^{(v_p)}D_{Y\}}^{(h)} = D_X^{(v_p)}D_Y^{(h)} - (-)^{|v_pXY|}D_Y^{(h)}D_X^{(v_p)},$$

$$D_{[X}^{(v_f)}D_{Y\}}^{(v_p)} = D_X^{(v_f)}D_Y^{(v_p)} - (-)^{|v_f X v_p Y|}D_Y^{(v_p)}D_X^{(v_f)}.$$

The local components of the ds-tensor fields (12) are introduced as follows:

$$R(\delta_K,\delta_J)\delta_H = R_H{}^I{}_{JK}\delta_I,\ R(\delta_K,\delta_J)\delta_{<B>} = R_{<B>.JK}^{\cdot<A>}\delta_{<A>}, \qquad (13)$$

$$R(\delta_{<C>},\delta_K)\delta_J = P_{JK<C>}^I\delta_I,\ R\left(\delta_{<C>},\delta_K\right)\delta_{<B>} = P_{<B>.K<C>}^{\cdot<A>}\delta_{<A>},$$

$$R(\delta_{<C>},\delta_{<B>})\delta_J = S_{J.<B><C>}^{\cdot I}\delta_I, R(\delta_{<D>},\delta_{<C>})\delta_{<B>} = S_{<B>.<C><D>}^{\cdot<A>}\delta_{<A>}.$$

Putting the components of a d-connection $D\Gamma = (L,...,L_{(p)},...,C,...,C_{(p)},...)$ in (13), by a direct computation, we obtain these locally adapted components of the curvature (curvatures):

$$R_H{}^I{}_{JK} = \delta_K L^I{}_{HJ} - (-)^{|KJ|}\delta_J L^I{}_{HK} +$$

$$L^M{}_{HJ}L^I{}_{MK} - (-)^{|KJ|}L^M{}_{HK}L^I{}_{MJ} + C^I{}_{H<A>}R^{<A>}{}_{JK},$$

$$R_{<B>\cdot JK}^{\cdot<A>} = \delta_K L^{<A>}{}_{<B>J} - (-)^{|KJ|}\delta_J L^{<A>}{}_{<B>K} +$$

$$L^{<C>}{}_{<B>J}L^{<A>}{}_{<C>K} - (-)^{|KJ|}L^{<C>}{}_{<B>K} + C^{<A>}{}_{<B><C>}R^{<C>}{}_{JK},$$

$$P_{J\cdot K<A>}^{\cdot I} = \delta_{<A>}L^I{}_{JK} - C^I{}_{J<A>|K} + C^I{}_{J<B>}P^{<B>}{}_{K<A>}, \qquad (14)$$

$$P_{<B>}{}^{<A>}{}_{K<C>} = \delta_{<C>}L^{<A>}{}_{<B>K} - C^{<A>}{}_{<B><C>|K} + C^{<A>}{}_{<B><D>}P^{<D>}{}_{K<C>},$$

$$S_{J\cdot<B><C>}^{\cdot I} = \delta_{<C>}C^I{}_{J<B>} - (-)^{|<B><C>|}\delta_{<B>}C^I{}_{J<C>} +$$

$$C^{<H>}{}_{J<B>}C^I{}_{<H><C>} - (-)^{|<B><C>|}C^{<H>}{}_{J<C>}C^I{}_{<H><B>},$$

$$S_{<B>}{}^{<A>}{}_{<C><D>} = \delta_{<D>}C^{<A>}{}_{<B><C>} - (-)^{|<C><D>|}\delta_{<C>}C^{<A>}{}_{<B><D>} +$$

$$C^{<E>}{}_{<B><C>}C^{<A>}{}_{<E><D>} - (-)^{|<C><D>|}C^{<E>}{}_{<B><D>}C^{<A>}{}_{<E><C>}.$$

Even and odd components of curvatures (14) can be written out by splitting indices into even and odd parts, for instance,

$$R_h{}^i{}_{jk} = \delta_k L^i{}_{hj} - \delta_j L^i{}_{hk} + L^m{}_{hj}L^i{}_{mk} - L^m{}_{hk}L^i{}_{mj} + C^i{}_{h<a>}R^{<a>}{}_{jk},$$

$$R_h{}^i{}_{j\hat{k}} = \delta_{\hat{k}}L^i{}_{hj} + \delta_j L^i{}_{h\hat{k}} + L^m{}_{hj}L^i{}_{m\hat{k}} + L^m{}_{h\hat{k}}L^i{}_{mj} + C^i{}_{h<a>}R^{<a>}{}_{j\hat{k}} \quad ,\ldots.$$

We omit explicit formulas for even–odd components because we shall not use them in this work.

5 Bianchi and Ricci Identities

The torsions and curvatures of every linear connection D on a vs-bundle $\tilde{\mathcal{E}}^{<z>}$ satisfy the following generalized Bianchi identities:

$$\sum_{SC}[(D_X T)(Y,Z) - R(X,Y)Z + T(T(X,Y),Z)] = 0,$$

$$\sum_{SC}[(D_X R)(U,Y,Z) + R(T(X,Y)Z)U] = 0, \tag{15}$$

where $\sum_{SC}$ means the respective supersymmretric cyclic sum over X, Y, Z and U. If D is a d-connection, then by using (11) and

$$v_p(D_X R)(U,Y,hZ) = 0,\ h(D_X R(U,Y,v_p Z) = 0, v_f(D_X R)(U,Y,v_p Z) = 0,$$

the identities (15) become

$$\sum_{SC}[h(D_X T)(Y,Z) - hR(X,Y)Z + hT(hT(X,Y),Z)+$$

$$hT(v_1 T(X,Y),Z) + ... + hT(v_z T(X,Y),Z)] = 0,$$

$$\sum_{SC}[v_f(D_X T)(Y,Z) - v_f R(X,Y)Z+$$

$$v_f T(hT(X,Y),Z)+\sum_{p\geq f} v_f T(v_p T(X,Y),Z)] = 0,$$

$$\sum_{SC}[h(D_X R)(U,Y,Z) + hR(hT(X,Y),Z)U+$$

$$hR(v_1 T(X,Y),Z)U + ... + hR(v_z T(X,Y),Z)U] = 0,$$

$$\sum_{SC}[v_f(D_X R)(U,Y,Z) + v_f R(hT(X,Y),Z)U+$$

$$\sum_{p\geq f} v_f R(v_p T(X,Y),Z)U] = 0. \tag{16}$$

In order to get the component form of these identities we insert correspondingly in (16) these values of triples (X,Y,Z), $(= (\delta_J, \delta_K, \delta_L)$, or $(\delta_{<D>}, \delta_{<C>}, \delta_{<B>}))$, and put successively $U = \delta_H$ and $U = \delta_{<A>}$. Taking into account (9),(10) and (12),(13) we obtain:

$$\sum_{SC[L,K,J\}}[T^I{}_{JK|H} + T^M{}_{JK}T^J{}_{HM} + R^{<A>}{}_{JK}C^I{}_{H<A>} - R_J{}^I{}_{KH}] = 0,$$

$$\sum_{SC[L,K,J\}}[R^{<A>}{}_{JK|H} + T^M{}_{JK}R^{<A>}{}_{HM} + R^{<B>}{}_{JK}P^{<A>}{}_{H<B>}] = 0,$$

$$C^I{}_{J<B>|K} - (-)^{|JK|} C^I{}_{K<B>|J} - T^I{}_{JK|<B>} + C^M{}_{J<B>} T^I{}_{KM} -$$

$$(-)^{|JK|} C^M{}_{K<B>} T^I{}_{JM} + T^M{}_{JK} C^I{}_{M<B>} + P^{<D>}{}_{J<B>} C^I{}_{K<D>}$$

$$-(-)^{|KJ|} P^{<D>}{}_{K<B>} C^I{}_{J<D>} + P_J{}^I{}_{K<B>} - (-)^{|KJ|} P_K{}^I{}_{J<B>} = 0,$$

$$P^{<A>}{}_{J<B>|K} - (-)^{|KJ|} P^{<A>}{}_{K<B>|J} - R^{<A>}{}_{JK\perp<B>} + C^M{}_{J<B>} R^{<A>}{}_{KM} -$$

$$(-)^{|KJ|} C^M{}_{K<B>} R^{<A>}{}_{JM} + T^M{}_{JK} P^{<A>}{}_{M<B>} + P^{<D>}{}_{J<B>} P^{<A>}{}_{K<D>} -$$

$$(-)^{|KJ|} P^{<D>}{}_{K<B>} P^{<A>}{}_{J<D>} - R^{<D>}{}_{JK} S^{<A>}{}_{B<D>} + R^{\cdot<A>}_{<B>\cdot JK} = 0,$$

$$C^I{}_{J<B>\perp<C>} - (-)^{|<B><C>|} C^I{}_{J<C>\perp<B>} + C^M{}_{J<C>} C^I{}_{M<B>} -$$

$$(-)^{|<B><C>|} C^M{}_{J<B>} C^I{}_{M<C>} + S^{<D>}{}_{<B><C>} C^I{}_{J<D>} - S^{\cdot I}_{J\cdot<B><C>} = 0,$$

$$P^{<A>}{}_{J<B>\perp<C>} - (-)^{|<B><C>|} P^{<A>}{}_{J<C>\perp<B>} +$$

$$S^{<A>}{}_{<B><C>|J} + C^M{}_{J<C>} P^{<A>}{}_{M<B>} -$$

$$(-)^{|<B><C>|} C^M{}_{J<B>} P^{<A>}{}_{M<C>} + P^{<D>}{}_{J<B>} S^{<A>}{}_{<C><D>} -$$

$$(-)^{|<C><B>|} P^{<D>}{}_{J<C>} S^{<A>}{}_{<B><D>} + S^{<D>}{}_{<B><C>} P^{<A>}{}_{J<D>} +$$

$$P_{<B>}{}^{<A>}{}_{J<C>} - (-)^{|<C><B>|} P_{<C>}{}^{<A>}{}_{J<B>} = 0,$$

$$\sum_{SC[<B>,<C>,<D>\}} [S^{<A>}{}_{<B><C>\perp<D>} +$$

$$S^{<F>}{}_{<B><C>} S^{<A>}{}_{<D><F>} - S_{<B>}{}^{<A>}{}_{<C><D>}] = 0,$$

$$\sum_{SC[H,J,L\}} [R_K{}^I{}_{HJ|L} - T^M{}_{HJ} R_K{}^I{}_{LM} - R^{<A>}{}_{HJ} P^{\cdot I}_{K\cdot L<A>}] = 0,$$

$$\sum_{SC[H,J,L\}} [R^{\cdot<A>}_{<D>\cdot HJ|L} - T^M{}_{HJ} R^{\cdot<A>}_{<D>\cdot LM} - R^{<C>}{}_{HJ} P_{<D>}{}^{<A>}{}_{L<C>}] = 0,$$

$$P^{\cdot I}_{K\cdot J<D>|L} - (-)^{|LJ|} P^{\cdot I}_{K\cdot L<D>|J} + R_K{}^I{}_{LJ\perp<D>} + C^M{}_{L<D>} R_K{}^I{}_{JM} -$$

$$(-)^{|LJ|} C^M{}_{J<D>} R_K{}^I{}_{LM} - T^M{}_{JL} P^{\cdot I}_{K\cdot M<D>} +$$

$$P^{<A>}{}_{L<D>} P^{\cdot I}_{K\cdot J<A>} - (-)^{|LJ|} P^{<A>}{}_{J<D>} P^{\cdot I}_{K\cdot L<A>} - R^{<A>}{}_{JL} S^{\cdot I}_{K\cdot<A><D>} = 0,$$

$$P_{<C>}{}^{<A>}{}_{J<D>|L} - (-)^{|LJ|} P_{<C>}{}^{<A>}{}_{L<D>|J} + R^{\cdot<A>}_{<C>\cdot LJ|<D>} +$$

$$C^M{}_{L<D>} R_{<C>}{}^{<A>}{}_{JM} - (-)^{|LJ|} C^M{}_{J<D>} R_{<C>}{}^{<A>}{}_{LM} -$$

$$T^M{}_{JL} P_{<C>}{}^{<A>}{}_{M<D>} + P^{<F>}{}_{L<D>} P_{<C>}{}^{<A>}{}_{J<F>} -$$

$$(-)^{|LJ|} P^{<F>}{}_{J<D>} P_{<C>}{}^{<A>}{}_{L<F>} - R^{<F>}{}_{JL} S_{<C>}{}^{<A>}{}_{F<D>} = 0,$$

$$P^{\cdot I}_{K\cdot J<D>\perp<C>} - (-)^{|<C><D>|} P^{\cdot I}_{K\cdot J<C>\perp<D>} + S_K{}^I{}_{<D><C>|J} +$$

$$C^{M}{}_{J<D>}P^{\cdot I}_{K\cdot M<C>} - (-)^{|<C><D>|}C^{M}{}_{J<C>}P^{\cdot I}_{K\cdot M<D>}+$$
$$P^{<A>}{}_{J<C>}S^{\cdot I}_{K\cdot <D><A>} - (-)^{|<C><D>|}P^{<A>}{}_{J<D>}S^{\cdot I}_{K\cdot <C><A>}+$$
$$S^{<A>}{}_{<C><D>}P^{\cdot I}_{K\cdot J<A>} = 0,$$
$$P_{<B>}{}^{<A>}{}_{J<D>\perp<C>} - (-)^{|<C><D>|}P_{<B>}{}^{<A>}{}_{J<C>\perp<D>} + S_{<B>}{}^{<A>}{}_{<C><D>|J}+$$
$$C^{M}{}_{J<D>}P_{<B>}{}^{<A>}{}_{M<C>} - (-)^{|<C><D>|}C^{M}{}_{J<C>}P_{<B>}{}^{<A>}{}_{M<D>}+$$
$$P^{<F>}{}_{J<C>}S_{<B>}{}^{<A>}{}_{<D><F>} - (-)^{|<C><D>|}P^{<F>}{}_{J<D>}S_{<B>}{}^{<A>}{}_{<C><F>}+$$
$$S^{<F>}{}_{<C><D>}P_{<B>}{}^{<A>}{}_{J<F>} = 0,$$
$$\sum_{SC[<B>,<C>,<D>\}}[S^{\cdot I}_{K.<B><C>\perp<D>} - S^{<A>}_{.<B><C>}S^{\cdot I}_{K.<D><A>}] = 0,$$
$$\sum_{SC[<B>,<C>,<D>\}}[S^{\cdot <A>}_{<F>.<B><C>\perp<D>} - S^{<E>}_{.<B><C>}S^{\cdot <A>}_{<F>.<E><A>}] = 0,$$

where $\sum\limits_{SC[<B>,<C>,<D>\}}$ is the supersymmetric cyclic sum over indices $<B>$, $<C>, <D>$.

As a consequence of a corresponding arrangement of (12) we obtain the Ricci identities (for simplicity we establish them only for ds-vector fields, although they may be written for every ds-tensor field):

$$D^{(h)}_{[X}D^{(h)}_{Y\}}hZ = R(hX,hY)hZ + D^{(h)}_{[hX,hY\}}hZ + \sum_{f=1}^{z} D^{(v_f)}_{[hX,hY\}}hZ, \tag{17}$$

$$D^{(v_p)}_{[X}D^{(h)}_{Y\}}hZ = R(v_pX,hY)hZ + D^{(h)}_{[v_pX,hY\}}hZ + \sum_{f=1}^{z} D^{(v_f)}_{[v_pX,hY\}}hZ,$$

$$D^{(v_p)}_{[X}D^{(v_P)}_{Y\}} = R(v_pX,v_pY)hZ + \sum_{f=1}^{z} D^{(v_f)}_{[v_pX,v_pY\}}hZ$$

and

$$D^{(h)}_{[X}D^{(h)}_{Y\}}v_pZ = R(hX,hY)v_pZ + D^{(h)}_{[hX,hY\}}v_pZ + \sum_{f=1}^{z} D^{(v_f)}_{[hX,hY\}}v_pZ, \tag{18}$$

$$D^{(v_f)}_{[X}D^{(h)}_{Y\}}v_pZ = R(v_fX,hY)v_pZ + \sum_{q=1}^{z} D^{(v_q)}_{[v_fX,hY\}}v_pZ + \sum_{q=1}^{z} D^{(v_q)}_{[v_fX,hY\}}v_pZ,$$

$$D^{(v_q)}_{[X}D^{(v_f)}_{Y\}}v_pZ = R(v_qX,v_fY)v_pZ + \sum_{s=1}^{z} D^{(v_s)}_{[v_fX,v_fY\}}v_pZ.$$

Putting $X = X^I(u)\frac{\delta}{\delta x^I} + X^{<A>}(u)\frac{\delta}{\partial y^{<A>}}$ and taking into account the local form of the h- and v-covariant s-derivatives and (9),(10),(12),(13) we can express respectively identities (17) and (18) in this form:

$$X^{<A>}{}_{|K|L} - (-)^{|KL|} X^{<A>}{}_{|L|K} =$$

$$R_{<B>}{}^{<A>}{}_{KL} X^{<B>} - T^{H}{}_{KL} X^{<A>}{}_{|H} - R^{<B>}{}_{KL} X^{<A>}{}_{\perp <B>},$$

$$X^{I}{}_{|K\perp <D>} - (-)^{|K<D>|} X^{I}{}_{\perp <D>|K} =$$

$$P^{\cdot I}_{H \cdot K<D>} X^H - C^{H}{}_{K<D>} X^{I}{}_{|H} - P^{<A>}{}_{K<D>} X^{I}{}_{\perp <A>},$$

$$X^{I}{}_{\perp <B> \perp <C>} - (-)^{|<B><C>|} X^{I}{}_{\perp <C> \perp <B>} =$$

$$S^{\cdot I}_{H \cdot <B><C>} X^H - S^{<A>}{}_{<B><C>} X^{I}{}_{\perp <A>}$$

and

$$X^{<A>}{}_{|K|L} - (-)^{|KL|} X^{<A>}{}_{|L|K} =$$

$$R_{<B>}{}^{<A>}{}_{KL} X^{<B>} - T^{H}{}_{KL} X^{<A>}{}_{|H} - R^{<B>}{}_{KL} X^{<A>}{}_{\perp <B>},$$

$$X^{<A>}{}_{|K\perp <B>} - (-)^{|<B>K|} X^{<A>}{}_{\perp <B>|K} =$$

$$P_{<B>}{}^{<A>}{}_{KC} X^C - C^{H}{}_{K<B>} X^{<A>}{}_{|H} - P^{<D>}{}_{K<B>} X^{<A>}{}_{\perp <D>},$$

$$X^{<A>}{}_{\perp <B> \perp <C>} - (-)^{|<C><B>|} X^{<A>}{}_{\perp <C> \perp <B>} =$$

$$S_{<D>}{}^{<A>}{}_{<B><C>} X^{<D>} - S^{<D>}{}_{<B><C>} X^{<A>}{}_{\perp <D>}.$$

We note that the above presented formulas generalize for higher order anisotropy the similar ones for locally anisotropic superspaces [28].

6 Cartan Structure Equations in DVS–Bundles

Let consider a ds-tensor field on $\tilde{\mathcal{E}}^{<z>}$:

$$t = t^I_{<A>} \delta_I \otimes \delta^{<A>}.$$

The d-connection 1-forms ω^I_J and $\tilde{\omega}^{<A>}_{<B>}$ are introduced as

$$Dt = (Dt^I_{<A>}) \delta_I \otimes \delta^{<A>}$$

with

$$Dt^I_{<A>} = dt^I_{<A>} + \omega^I_J t^J_{<A>} - \tilde{\omega}^{<B>}_{<A>} t^I_{<B>} = t^I_{<A>|J} dx^J + t^I_{<A>\perp <B>} \delta y^{<B>}.$$

For the d-connection 1-forms of a d-connection D on $\tilde{\mathcal{E}}^{<z>}$ defined by ω_J^I and $\tilde{\omega}_{<B>}^{<A>}$ one holds the following structure equations:

$$d(d^I) - d^H \wedge \omega_H^I = -\Omega,\ d(\delta^{<A>}) - \delta^{<B>} \wedge \tilde{\omega}_{<B>}^{<A>} = -\tilde{\Omega}^{<A>},$$

$$d\omega_J^I - \omega_J^H \wedge \omega_H^I = -\Omega_J^I,\ d\tilde{\omega}_{<B>}^{<A>} - \tilde{\omega}_{<B>}^{<C>} \wedge \tilde{\omega}_{<C>}^{<A>} = -\tilde{\Omega}_{<B>}^{<A>},$$

in which the torsion 2-forms Ω^I and $\tilde{\Omega}^{<A>}$ are given respectively by formulas:

$$\Omega^I = \frac{1}{2} T^I{}_{JK} d^J \wedge d^K + \frac{1}{2} C^I{}_{J<C>} d^J \wedge \delta^{<C>},$$

$$\tilde{\Omega}^{<A>} = \frac{1}{2} R^{<A>}{}_{JK} d^J \wedge d^K + \frac{1}{2} P^{<A>}{}_{J<C>} d^J \wedge \delta^{<C>} + \frac{1}{2} S^{<A>}{}_{<B><C>} \delta^{<B>} \wedge \delta^{<C>},$$

and

$$\Omega_J^I = \frac{1}{2} R_J{}^I{}_{KH} d^K \wedge d^H + \frac{1}{2} P_{J\cdot K<C>}^{\cdot I} d^K \wedge \delta^{<C>} + \frac{1}{2} S_{J\cdot K<C>}^{\cdot I} \delta^{<B>} \wedge \delta^{<C>},$$

$$\tilde{\Omega}_{<B>}^{<A>} = \frac{1}{2} R_{<B>\cdot KH}^{\cdot <A>} d^K \wedge d^H +$$

$$\frac{1}{2} P_{<B>}{}^{<A>}{}_{K<C>} d^K \wedge \delta^{<C>} + \frac{1}{2} S_{<B>}{}^{<A>}{}_{<C><D>} \delta^{<C>} \wedge \delta^{<D>}.$$

We have defined the exterior product on s-space as to satisfy the property

$$\delta^{<\alpha>} \wedge \delta^{<\beta>} = -(-)^{|<\alpha><\beta>|} \delta^{<\beta>} \wedge \delta^{<\alpha>}.$$

7 Metrics in DVS–Bundles

The base $\tilde{M}$ of dvs-bundle $\tilde{\mathcal{E}}^{<z>}$is considered to be a connected and paracompact s-manifold.

A metric structure on the total space $\tilde{E}^{<z>}$ of a dvs-bundle $\tilde{\mathcal{E}}^{<z>}$is a supersymmetric, second order, covariant s-tensor field

$$G = G_{<\alpha><\beta>} \partial^{<\alpha>} \otimes \partial^{<\beta>}$$

which in every point $u \in \tilde{\mathcal{E}}^{<z>}$ is given by nondegenerate supersymmetric matrix $G_{<\alpha><\beta>} = G(\partial_{<\alpha>}, \partial_{<\beta>})$ (with nonvanishing superdeterminant, $sdetG \neq 0$).

The metric and N-connection structures on $\tilde{\mathcal{E}}^{<z>}$ are compatible if there are satisfied conditions:

$$G(\delta_I, \partial_{<A>}) = 0, G(\delta_{A_f}, \partial_{A_p}) = 0,\ z \geq p > f \geq 1,$$

or, in consequence,

$$G_{I<A>} - N_I^{<B>} h_{<A><B>} = 0, G_{A_f A_p} - N_{A_f}^{B_p} h_{A_p B_p} = 0, \quad (19)$$

where

$$G_{I<A>} = G(\partial_I, \partial_{<A>}), G_{A_f A_p} = G(\partial_{A_f}, \partial_{A_p}).$$

From (42) one follows

$$N_I^{<B>} = h^{<B><A>} G_{I<A>},\ N_{A_f}^{A_p} = h^{A_p B_p} G_{A_f B_p}, ...,$$

where matrices $h^{<A><B>}, h^{A_p B_p}, ...$ are respectively s–inverse to matrices

$$h_{<A><B>} = G(\partial_{<A>}, \partial_{<B>}), h_{A_p B_p} = G(\partial_{A_p}, \partial_{B_p}).$$

So, in this case, the coefficients of N-connection are uniquely determined by the components of the metric on $\tilde{\mathcal{E}}^{<z>}$.

A compatible with N–connection metric on $\tilde{\mathcal{E}}^{<z>}$ is written in irreducible form as

$$G(X,Y) = G(hX, hY) + G(v_1 X, v_1 Y) + ... + G(v_z X, v_z Y), \quad X, Y \in \Xi(\tilde{\mathcal{E}}^{<z>}),$$

and looks locally as

$$G = g_{\alpha\beta}(u) \delta^\alpha \otimes \delta^\beta = g_{IJ} d^I \otimes d^J + h_{<A><B>} \delta^{<A>} \otimes \delta^{<B>} =$$

$$g_{IJ} d^I \otimes d^J + h_{A_1 B_1} \delta^{A_1} \otimes \delta^{B_1} + h_{A_2 B_2} \delta^{A_2} \otimes \delta^{B_2} + ... + h_{A_z B_z} \delta^{A_z} \otimes \delta^{B_z}. \quad (20)$$

A d-connection D on $\tilde{\mathcal{E}}^{<z>}$ is metric, or compatible with metric G, if conditions

$$D_{<\alpha>} G_{<\beta><\gamma>} = 0$$

are satisfied.

A d-connection D on $\tilde{\mathcal{E}}^{<z>}$ provided with a metric G is a metric d-connection if and only if

$$D_X^{(h)}(hG) = 0, D_X^{(h)}(v_p G) = 0, D_X^{(v_p)}(hG) = 0, D_X^{(v_f)}(v_p G) = 0 \quad (21)$$

for every $, f, p = 1, 2, ..., z,$ and $X \in \Xi(\tilde{\mathcal{E}}^{<z>})$. Conditions (21) are written in locally adapted form as

$$g_{IJ|K} = 0, g_{IJ\perp<A>} = 0, h_{<A><B>|K} = 0, h_{<A><B>\perp<C>} = 0.$$

In every dvs–bundle provided with compatible N–connection and metric structures one exists a metric d-connection (called the canonical d-connection

associated to G) depending only on components of G-metric and N-connection. Its local coefficients $C\Gamma = (\grave{L}^{I}_{JK}, \grave{L}^{<A>}_{<B>K}, \grave{C}^{I}_{J<C>}, \grave{C}^{<A>}_{<B><C>})$ are as follows:

$$\grave{L}^{I}_{JK} = \frac{1}{2} g^{IH} (\delta_K g_{HJ} + \delta_J g_{HK} - \delta_H g_{JK}), \qquad (22)$$

$$\grave{L}^{<A>}_{<B>K} = \delta_{<B>} N^{<A>}_{K} +$$

$$\frac{1}{2} h^{<A><C>} [\delta_{<K>} h_{<B><C>} - (\delta_{<B>} N^{<D>}_{K}) h_{<\dot{D}><C>} - (\delta_{<C>} N^{<D>}_{K}) h_{<\dot{D}><B>}],$$

$$\grave{C}^{I}_{J<C>} = \frac{1}{2} g^{IK} \delta_{<C>} g_{JK},$$

$$\grave{C}^{<A>}_{<B><C>} = \frac{1}{2} h^{<A><D>} (\delta_{<C>} h_{<D><B>} + \delta_{<B>} h_{<D><C>} - \delta_{<D>} h_{<B><C>}.$$

We emphasize that, in general, the torsion of $C\Gamma$–connection (22) does not vanish.

It should be noted here that on dvs-bundles provided with N-connection and d-connection and metric really it is defined a multiconnection ds-structure, i.e. we can use in an equivalent geometric manner different types of d- connections with various properties. For example, for modeling of some physical processes we can use an extension of the Berwald d–connection

$$B\Gamma = (L^{I}{}_{JK}, \delta_{<B>} N^{<A>}_{K}, 0, C^{<A>}_{<B><C>}), \qquad (23)$$

where $L^{I}{}_{JK} = \grave{L}^{I}_{JK}$ and $C^{<A>}{}_{<B><C>} = \grave{C}^{<A>}_{<B><C>}$, which is hv-metric, i.e. satisfies conditions:

$$D^{(h)}_{X} hG = 0, ..., D^{(v_p)}_{X} v_p G = 0, ..., D^{(v_z)}_{X} v_z G = 0$$

for every $X \in \Xi(\tilde{\mathcal{E}}^{<z>})$, or in locally adapted coordinates,

$$g_{IJ|K} = 0, h_{<A><B>\perp<C>} = 0.$$

As well we can introduce the Levi-Civita connection

$$\{\frac{<\alpha>}{<\beta><\gamma>}\} = \frac{1}{2} G^{<\alpha><\beta>} (\partial_{<\beta>} G_{<\tau><\gamma>} + \partial_{<\gamma>} G_{<\tau><\beta>} - \partial_{<\tau>} G_{<\beta><\gamma>}),$$

constructed as in the Riemann geometry from components of metric $G_{<\alpha><\beta>}$ by using partial derivations $\partial_{<\alpha>} = \frac{\partial}{\partial u^{<\alpha>}} = (\frac{\partial}{\partial x^I}, \frac{\partial}{\partial y^{<A>}})$, which is metric but not a d-connection.

In our further considerations we shall largely use the Christoffel d–symbols defined similarly as components of Levi-Civita connection but by using la-partial derivations,

$$\tilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} = \frac{1}{2} G^{<\alpha><\tau>} (\delta_{<\beta>} G_{<\tau><\gamma>} + \delta_{<\gamma>} G_{<\tau><\beta>} - \delta_{<\tau>} G_{<\beta><\gamma>}), \quad (24)$$

having components

$$C\tilde{\Gamma} = (L^I{}_{JK}, 0, 0, C^{<A>}{}_{<B><C>}),$$

where coefficients $L^I{}_{JK}$ and $C^{<A>}{}_{<B><C>}$ must be computed as in formulas (20).

We can express arbitrary d-connection as a deformation of the background d-connection (23):

$$\Gamma^{<\alpha>}{}_{<\beta><\gamma>} = \tilde{\Gamma}^{<\alpha>}_{\cdot<\beta><\gamma>} + P^{<\alpha>}{}_{<\beta><\gamma>}, \tag{25}$$

where $P^{<\alpha>}{}_{<\beta><\gamma>}$ is called the deformation ds-tensor. Putting splitting (25) into (19) and (23) we can express torsion $T^{<\alpha>}{}_{<\beta><\gamma>}$ and curvature $R_{<\beta>}{}^{<\alpha>}{}_{<\gamma><\delta>}$ of a d-connection $\Gamma^{<\alpha>}{}_{<\beta><\gamma>}$ as respective deformations of torsion $\tilde{T}^{<\alpha>}_{<\beta><\gamma>}$ and torsion $\tilde{R}^{\cdot<\alpha>}_{<\beta>\cdot<\gamma><\delta>}$ for connection $\tilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>}$:

$$T^{<\alpha>}{}_{<\beta><\gamma>} = \tilde{T}^{<\alpha>}_{\cdot<\beta><\gamma>} + \ddot{T}^{<\alpha>}_{\cdot<\beta><\gamma>}$$

and

$$R_{<\beta>}{}^{<\alpha>}{}_{<\gamma><\delta>} = \tilde{R}^{\cdot<\alpha>}_{<\beta>\cdot<\gamma><\delta>} + \ddot{R}^{\cdot<\alpha>}_{<\beta>\cdot<\gamma><\delta>},$$

where

$$\tilde{T}^{<\alpha>}_{<\beta><\gamma>} = \tilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} - (-)^{|<\beta><\gamma>|}\tilde{\Gamma}^{<\alpha>}_{<\gamma><\beta>} + w^{<\alpha>}{}_{<\gamma><\delta>},$$

$$\ddot{T}^{<\alpha>}_{<\beta><\gamma>} = \ddot{\Gamma}^{<\alpha>}_{<\beta><\gamma>} - (-)^{|<\beta><\gamma>|}\ddot{\Gamma}^{<\alpha>}_{<\gamma><\beta>},$$

and

$$\tilde{R}^{\cdot<\alpha>}_{<\beta>\cdot<\gamma><\delta>} = \delta_{<\delta>}\tilde{\Gamma}^{<\alpha>}_{<\beta><\gamma>} - (-)^{|<\gamma><\delta>|}\delta_{<\gamma>}\tilde{\Gamma}^{<\alpha>}_{<\beta><\delta>} +$$

$$\tilde{\Gamma}^{<\varphi>}_{<\beta><\gamma>}\tilde{\Gamma}^{<\alpha>}_{<\varphi><\delta>} - (-)^{|<\gamma><\delta>|}\tilde{\Gamma}^{<\varphi>}_{<\beta><\delta>}\tilde{\Gamma}^{<\alpha>}_{<\varphi><\gamma>} + \tilde{\Gamma}^{<\alpha>}_{<\beta><\varphi>}w^{<\varphi>}{}_{<\gamma><\delta>},$$

$$\ddot{R}^{\cdot<\alpha>}_{<\beta>\cdot<\gamma><\delta>} = \tilde{D}_{<\delta>}P^{<\alpha>}{}_{<\beta><\gamma>} - (-)^{|<\gamma><\delta>|}\tilde{D}_{<\gamma>}P^{<\alpha>}{}_{<\beta><\delta>} +$$

$$P^{<\varphi>}{}_{<\beta><\gamma>}P^{<\alpha>}{}_{<\varphi><\delta>} - (-)^{|<\gamma><\delta>|}P^{<\varphi>}{}_{<\beta><\delta>}P^{<\alpha>}{}_{<\varphi><\gamma>}$$

$$+P^{<\alpha>}{}_{<\beta><\varphi>}w^{<\varphi>}{}_{<\gamma><\delta>},$$

the nonholonomy coefficients $w^{<\alpha>}{}_{<\beta><\gamma>}$ are defined as

$$[\delta_{<\alpha>}, \delta_{<\beta>}\} = \delta_{<\alpha>}\delta_{<\beta>} - (-)^{|<\alpha><\beta>|}\delta_{<\beta>}\delta_{<\alpha>} = w^{<\tau>}{}_{<\alpha><\beta>}\delta_{<\tau>}.$$

We emphasize that if from geometric point of view all considered d–connections are "equal in rights", the construction of physical models on la-spaces requires an explicit fixing of the type of d–connection and metric structures.

8 Higher Order Tangent S–Bundles

The aim of this section is to present a study of supersymmetric extensions from $\widetilde{M}$ to $T\tilde{M}$ and $Osc^{(z)}\widetilde{M}$ and to consider corresponding prolongations of Riemann and generalized Finsler structures (on classical and new approaches to Finsler geometry, its generalizations and applications in physics se, for example, [13, 10, 25, 19, 20, 3, 4, 18, 22, 1, 2, 7, 6]).

The presented in the previous sections basic results on dvs-bundles $\tilde{\mathcal{E}}^{<z>}$ provided with N-connection, d-connection and metric structures can be correspondingly adapted to the osculator s–bundle $\left(Osc^z\tilde{M}, \pi, \tilde{M}\right)$. In this case the dimension of the base space and typical higher orders fibre coincides and we shall not distinguish indices of geometrical objects.

Coefficients of a d–connection $D\Gamma(N) = (L^I_{JM}, C^I_{(1)JM}, ..., C^I_{(z)JM})$ in $Osc^z\tilde{M}$, with respect to a la–base are introduced as to satisfy equations

$$D_{\frac{\delta}{\delta x^I}}\frac{\delta}{\delta y^I_{(f)}} = L^M_{IJ}\frac{\delta}{\delta y^M_{(f)}},\ D_{\frac{\delta}{\delta y^J_{(p)}}}\frac{\delta}{\delta y^I_{(f)}} = C^M_{(p)IJ}\frac{\delta}{\delta y^M_{(f)}}, \tag{26}$$

$$(f = 0, 1, ..., z; p = 1, ..., z, \text{ and } \ y^I_{(0)} = x^I).$$

A metric structure on $Osc^z\tilde{M}$ is ds-tensor s–symmetric field $g_{IJ}(u_{(z)}) = g_{IJ}(x, y_{(1)}, y_{(2)}, ..., y_{(z)})$ of type $(0,2), srank|g_{ij}| = (n,m)$. The N–lift of Sasaki type of g_{IJ} is given by (see (20)) defines a global Riemannian s–structure (if $\widetilde{M}$ is a s–differentiable, paracompact s-manifold):

$$G = g_{IJ}(u_{(z)})dx^I \otimes dx^J + g_{IJ}(u_{(z)})dy^I_{(1)} \otimes dy^J_{(1)} + ... + g_{IJ}(u_{(z)})dy^I_{(z)} \otimes dy^J_{(z)}. \tag{27}$$

The condition of compatibility of a d–connection (26) with metric (27) is expressed as

$$D_X G = 0, \forall X \in \Xi(Osc^z\tilde{M}),$$

or, by using d–covariant partial derivations $|_{(p)}$ defined by coefficients $(L^I_{JM}, C^I_{(1)JM}, ..., C^I_{(z)JM})$,

$$g_{IJ|M=0},\ g_{IJ|_{(p)M}} = 0, (p = 1, ..., z).$$

An example of compatible with metric d–connection is given by Christoffel d-symbols (see (24)):

$$L^M_{IJ} = \frac{1}{2}g^{MK}\left(\frac{\delta g_{KJ}}{\partial x^I} + \frac{\delta g_{IK}}{\partial x^J} - \frac{\delta g_{IJ}}{\partial x^K}\right),$$

$$C^M_{(p)IJ} = \frac{1}{2}g^{MK}\left(\frac{\delta g_{KJ}}{\partial y^I_{(p)}} + \frac{\delta g_{IK}}{\partial y^J_{(p)}} - \frac{\delta g_{IJ}}{\partial y^K_{(p)}}\right); p = 1, 2, ..., z.$$

9 Supersymmetric Extensions of Finsler Spaces

We start our considerations with the ts-bundle $\tilde{TM}$. An s-vector $X \in \Xi(\tilde{TM})$ is decomposed with respect to la–bundles as

$$X = X(u)^I \delta_I + Y(u)^I \partial_I,$$

where $u = u^\alpha = (x^I, y^J)$ local coordinates. The s–tangent structures (6) are transformed into a global map

$$J : \Xi(\tilde{TM}) \rightarrow \Xi(\tilde{TM})$$

which does not depend on N-connection structure:

$$J(\frac{\delta}{\delta x^I}) = \frac{\partial}{\partial y^I}$$

and

$$J(\frac{\partial}{\partial y^I}) = 0.$$

This endomorphism is called the natural (or canonical) almost tangent structure on $\tilde{TM}$; it has the properties:

$$1) J^2 = 0, \quad 2) ImJ = KerJ = V\tilde{TM}$$

and 3) the Nigenhuis s-tensor,

$$N_J(X,Y) = [JX, JY\} - J[JX, Y\} - J[X, JY]$$

$$(X, Y \in \Xi(TN))$$

identically vanishes, i.e. the natural almost tangent structure J on $\tilde{TM}$ is integrable.

A generalized Lagrange superspace, GLS– space, is a pair $GL^{n,m} = (\tilde{M}, g_{IJ}(x,y))$, where $g_{IJ}(x,y)$ is a ds– tensor field on $\tilde{\tilde{TM}} = \tilde{TM} - \{0\}$, s–symmetric of superrank (n,m).

We call g_{IJ} as the fundamental ds-tensor, or metric ds-tensor, of GLS-space.

There exists an unique d-connection $C\Gamma(N)$ which is compatible with $g_{IJ}(u)$ and has vanishing torsions $T^I{}_{JK}$ and $S^I{}_{JK}$ (see formulas (26) rewritten for ts-bundles). This connection, depending only on $g_{IJ}(u)$ and $N^I_J(u)$ is called the canonical metric d-connection of GLS-space. It has coefficients

$$L^I{}_{JK} = \frac{1}{2} g^{IH} (\delta_J g_{HK} + \delta_H g_{JK} - \delta_H g_{JK}),$$

$$C^I{}_{JK} = \frac{1}{2} g^{IH} (\partial_J g_{HK} + \partial_H g_{JK} - \partial_H g_{JK}).$$

There is a unique normal d-connection $D\Gamma(N) = (\bar{L}^I_{\cdot JK}, \bar{C}^I_{\cdot JK})$ which is metric and has a priori given torsions $T^I{}_{JK}$ and $S^I{}_{JK}$. The coefficients of $D\Gamma(N)$ are the following ones:

$$\bar{L}^I_{\cdot JK} = L^I{}_{JK} - \frac{1}{2} g^{IH} (g_{JR} T^R{}_{HK} + g_{KR} T^R{}_{HJ} - g_{HR} T^R{}_{KJ}),$$

$$\bar{C}^I_{\cdot JK} = C^I{}_{JK} - \frac{1}{2} g^{IH} (g_{JR} S^R{}_{HK} + g_{KR} S^R{}_{HJ} - g_{HR} S^R{}_{KJ}),$$

where $L^I{}_{JK}$ and $C^I{}_{JK}$ are the same as for the $C\Gamma(N)$–connection (26).

The Lagrange spaces were introduced [14] in order to geometrize the concept of Lagrangian in mechanics (the Lagrange geometry is studied in details in [19, 20]). For s-spaces we present this generalization:

A Lagrange s-space, LS-space, $L^{n,m} = (\tilde{M}, g_{IJ})$, is defined as a particular case of GLS-space when the ds-metric on $\tilde{M}$ can be expressed as

$$g_{IJ}(u) = \frac{1}{2} \frac{\partial^2 L}{\partial y^I \partial y^J}, \tag{28}$$

where $L : T\tilde{M} \to \Lambda$, is a s-differentiable function called a s-Lagrangian on $\tilde{M}$.

Now we consider the supersymmetric extension of the Finsler space:

A Finsler s-metric on $\tilde{M}$ is a function $F_S : T\tilde{M} \to \Lambda$ having the properties:

1. The restriction of F_S to $\tilde{T\tilde{M}} = T\tilde{M} \setminus \{0\}$ is of the class G^∞ and F is only supersmooth on the image of the null cross–section in the ts-bundle to $\tilde{M}$.

2. The restriction of F to $\tilde{T\tilde{M}}$ is positively homogeneous of degree 1 with respect to (y^I), i.e. $F(x, \lambda y) = \lambda F(x, y)$, where λ is a real positive number.

3. The restriction of F to the even subspace of $\tilde{T\tilde{M}}$ is a positive function.

4. The quadratic form on $\Lambda^{n,m}$ with the coefficients

$$g_{IJ}(u) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^I \partial y^J}$$

defined on $\tilde{T\tilde{M}}$ is nondegenerate.

A pair $F^{n,m} = (\tilde{M}, F)$ which consists from a supersmooth s-manifold $\tilde{M}$ and a Finsler s-metric is called a Finsler superspace, FS-space.

It's obvious that FS-spaces form a particular class of LS-spaces with s-Lagrangian $L = F^2$ and a particular class of GLS-spaces with metrics of type (28).

For a FS-space we can introduce the supersymmetric variant of nonlinear Cartan connection [10, 25] :

$$N_J^I(x,y) = \frac{\partial}{\partial y^J} G^{*I},$$

where

$$G^{*I} = \frac{1}{4} g^{*IJ} \left(\frac{\partial^2 \varepsilon}{\partial y^I \partial x^K} y^K - \frac{\partial \varepsilon}{\partial x^J} \right), \quad \varepsilon(u) = g_{IJ}(u) y^I y^J,$$

and g^{*IJ} is inverse to $g^*_{IJ}(u) = \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial y^I \partial y^J}$. In this case the coefficients of canonical metric d-connection (26) gives the supersymmetric variants of coefficients of the Cartan connection of Finsler spaces. A similar remark applies to the Lagrange superspaces.

10 Higher Order Prolongations of Finsler and Lagrange S–Spaces

The geometric constructions on $T\widetilde{M}$ from the previous subsection have corresponding generalizations to the $Osc^{(z)}\widetilde{M}$ s–bundle. The basic idea is similar to that used for prolongations of geometric structures (see [23] for prolongations on tangent bundle). Having defined a metric structure $g_{IJ}(x)$ on a s-manifold $\widetilde{M}$ we can extend it to the $Osc^z\tilde{M}$ s–bundle by considering $g_{IJ}(u_{(z)}) = g_{IJ}(x)$ in (27). R. Miron and Gh. Atanasiu [21] solved the problem of prolongations of Finsler and Lagrange structures on osculator bundle. In this subsection we shall analyze supersymmetric extensions of Finsler and Lagrange structures as well present a brief introduction into geometry of higher order Lagrange s-spaces.

Let $F^{n,m} = (\tilde{M}, F)$ be a FS–space with the fundamental function $F_S : T\tilde{M} \to \Lambda$ on $\widetilde{M}$. A prolongation of F on $Osc^z\tilde{M}$ is given by a map

$$(F \circ \pi_1^z)(u_{(z)}) = F(u_{(1)})$$

and corresponding fundamental tensor

$$g_{IJ}(u_{(1)}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^I_{(1)} \partial y^J_{(1)}},$$

for which

$$(g_{IJ} \circ \pi_1^z)(u_{(z)}) = g_{IJ}(u_{(1)}).$$

So, $g_{IJ}(u_{(1)})$ is a ds–tensor on
$\widetilde{Osk^z\widetilde{M}} = Osc^z\tilde{M}/\{0\} = \{(u_{(z)}) \in Osc^z\tilde{M}, srank|y^I_{(1)}| = 1\}.$

The Christoffel symbols

$$\gamma^{M}_{IJ}(u^{(1)}) = \frac{1}{2} g^{MK}(u_{(1)})(\frac{\partial g_{KI}(u_{(1)})}{\partial x^J} + \frac{\partial g_{JK}(u_{(1)})}{\partial x^I} - \frac{\partial g_{IJ}(u_{(1)})}{\partial x^K})$$

define the Cartan nonlinear connection [9]:

$$G^{I}_{(N)J} = \frac{1}{2} \frac{\partial}{\partial y^{J}_{(1)}} (\gamma^{I}_{KM} y^{K}_{(1)} y^{M}_{(1)}). \quad (29)$$

The dual coefficients for the N-connection (see formulas (5)) are recurrently computed by using (29) and operator

$$\Gamma = y^{I}_{(1)} \frac{\partial}{\partial x^I} + 2\ y^{I}_{(2)} \frac{\partial}{\partial y^{I}_{(1)}} + ... + z\ y^{I}_{(z)} \frac{\partial}{\partial y^{I}_{(z-1)}},$$

$$M^{I}_{(1)J} = G^{I}_{(N)J},$$

$$M^{I}_{(2)J} = \frac{1}{2}[\Gamma G^{I}_{(N)J} + G^{I}_{(N)K} M^{K}_{(1)J}],$$

....

$$M^{I}_{(z)J} = \frac{1}{z}[\Gamma M^{I}_{(z-1)J} + G^{I}_{(N)K} M^{K}_{(z-1)J}].$$

The prolongations of FS–spaces can be generalized for Lagrange s–spaces (on Lagrange spaces and theirs higher order extensions see [19, 20, 21] and on supersymmetric extensions of Finsler geometry see [28]). Let $L^{n,m} = (\tilde{M}, g_{IJ})$ be a Lagrange s–space. The Lagrangian $L : T\widetilde{M} \to \Lambda$ can be extended on $Osc^z \tilde{M}$ by using maps of the Lagrangian, $(L \circ \pi^z_1)(u_{(z)}) = L\left(u_{(1)}\right)$, and, as a consequence, of the fundamental tensor (28), $(g_{IJ} \circ \pi^z_1)(u_{(z)}) = g_{IJ}\left(u_{(1)}\right)$.

We introduce the notion of Lagrangian of z–order on a differentiable s–manifold $\widetilde{M}$ as a map $L^z : Osc^z \tilde{M} \to \Lambda$. In order to have concordance with the definitions proposed by [21] we require the even part of the fundamental ds–tensor to be of constant signature. Here we also note that questions to considered in this subsection, being an supersymmetric approach, are connected with the problem of elaboration of the so–called higher order analytic mechanics (see, for instance, [12, 17, 16, 26]).

A Lagrangian s–differentiable of order z ($z = 1, 2, 3, ...$) on s-differentiable s–manifold $\widetilde{M}$ is an application $L^{(z)} : Osc^z \widetilde{M} \to \Lambda$, s–differentiable on $\widetilde{Osk^z}\widetilde{M}$ and smooth in the points of $Osc^z \widetilde{M}$ where $y^{I}_{(1)} = 0$.

It is obvious that

$$g_{IJ}(x, y_{(1)}, ..., y_{(z)}) = \frac{1}{2} \frac{\partial^2 L^{(z)}}{\partial y^{I}_{(z)} \partial y^{J}_{(z)}}$$

is a ds–tensor field because with respect to coordinate transforms (1) one holds transforms

$$K_I^{I'} K_J^{J'} g_{I'J'} = g_{IJ.}$$

A Lagrangian L is regular if $srank|g_{IJ}| = (n, m)$.

A Lagrange s–space of z–order is a pair $L^{(z,n,m)} = (\widetilde{M}, L^{(z)})$, where $L^{(z)}$ is a s–differentiable regular Lagrangian of z–order, and with ds–tensor g_{IJ} being of constant signature on the even part of the basic s–manifold.

For details on nonsupersymmetric osculator bundles see [21].

11 Conclusions and Discussion

In the present work we focused on the definition and calculation of basic geometric structures on superspaces with higher order generic anisotropy. In the framework of such spaces it seems more convenient to touch on the problem of formulation of higher dimensional and locally anisotropic classical and quantum field theories. It should be noted here that the general approach of modeling of locally anisotropic geometries (Finsler, Lagrange and various higher order extensions and prolongations; we are very much inspired by R. Miron, M. Anastasiei and G. Atanasiu works [19, 20, 21]) made apparent the possibility and manner of formulation of physical theories by adapting geometric and physical constructions to the N–connection structure. Former considerations based on straightforward applications of Finsler spaces are characterized by cumbersome tensorial calculations and different ambiguities in physical interpretation. In our case we are dealing with a standard (super)bundle technique which allow us a geometric study being very similar to that for supersymmetic extensions of Einstein–Cartan spaces with torsion but additionally provided with a N–connection structure. From the viewpoint of modern Kaluza–Klain theories the N–connection can be considered as a "reduction" field from higher dimensions to lower dimensional ones. Higher order anisotropies can be treated in this case as relic interactions and (classic or quantum) fluctuations from higher dimensions. This is a matter of our further investigations (see also our previous works [27, 28, 30]).

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